

Saari's homographic conjecture for planar equal-mass three-body problem in Newton gravity

Toshiaki Fujiwara¹, Hiroshi Fukuda², Hiroshi Ozaki³
and Tetsuya Taniguchi⁴

^{1 2 4}College of Liberal Arts and Sciences, Kitasato University, 1-15-1 Kitasato, Sagami-hara, Kanagawa 252-0329, Japan

³General Education Program Center, Tokai University, Shimizu Campus, 3-20-1, Orido, Shimizu, Shizuoka 424-8610, Japan

E-mail: ¹fujiiwara@kitasato-u.ac.jp, ²fukuda@kitasato-u.ac.jp,
³ozaki@tokai-u.jp, ⁴tetsuya@kitasato-u.ac.jp

Abstract. Saari's homographic conjecture in N -body problem under the Newton gravity is the following; configurational measure $\mu = \sqrt{I}U$, which is the product of square root of the moment of inertia $I = (\sum m_k)^{-1} \sum m_i m_j r_{ij}^2$ and the potential function $U = \sum m_i m_j / r_{ij}$, is constant if and only if the motion is homographic. Where m_k represents mass of body k and r_{ij} represents distance between bodies i and j . We prove this conjecture for planar equal-mass three-body problem.

In this work, we use three sets of shape variables. In the first step, we use $\zeta = 3q_3/(2(q_2 - q_1))$ where $q_k \in \mathbb{C}$ represents position of body k . Using $r_1 = r_{23}/r_{12}$ and $r_2 = r_{31}/r_{12}$ in intermediate step, we finally use μ itself and $\rho = I^{3/2}/(r_{12}r_{23}r_{31})$. The shape variables μ and ρ make our proof simple.

PACS numbers: 45.20.D-, 45.20.Jj, 45.50.Jf

Submitted to: *J. Phys. A: Math. Gen.*

1. Saari's homographic conjecture

In 2005, Donald Saari formulated his conjecture in the following form [10, 11]; in the N -body problem under the potential function

$$U = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{r_{ij}^\alpha}, \quad \alpha > 0, \quad (1)$$

a motion has a constant configurational measure

$$\mu = I^{\alpha/2} U \quad (2)$$

if and only if the motion is homographic. Here, r_{ij} represents the mutual distance between the bodies i and j , and I represents the moment of inertia

$$I = \left(\sum_{1 \leq k \leq N} m_k \right)^{-1} \sum_{1 \leq i < j \leq N} m_i m_j r_{ij}^2. \quad (3)$$

Florin Diacu, Toshiaki Fujiwara, Ernesto Perez-Chavela and Manuele Santoprete called this conjecture the ‘‘Saari’s homographic conjecture’’ and partly proved this conjecture for some cases [2]. Recently, the present authors proved this conjecture for planar equal-mass three-body problem for $\alpha = 2$ [3]. In this paper, we extend our proof to $\alpha = 1$, the Newton gravity.

In section 2, we derive the equations of motion for the size change, rotation and shape change. To do this, we use the shape variable ζ ,

$$\zeta = \frac{3}{2} \frac{q_3}{q_2 - q_1}, \quad (4)$$

introduced by Richard Moeckel and Richard Montgomery [8]. Here, $q_k \in \mathbb{C}$, $k = 1, 2, 3$ represents position of the body k . Then, in the section 3, we investigate motions with $\mu = \text{constant}$ and non-homographic, and we derive a necessary condition that must be satisfied by such motion. The contents in the sections 2 and 3 are review of our previous paper [3], although we changed few notations. To prove the Saari’s conjecture, we will show that no finite orbit satisfies the necessary condition. To attain this purpose, the expression of the necessary condition by ζ is too complex. To simplify the expression, we will use other set of shape variables,

$$r_1 = |\zeta - 1/2| = r_{23}/r_{12}, \quad r_2 = |\zeta + 1/2| = r_{31}/r_{12}. \quad (5)$$

Then, using the invariance of the system under the permutations of $\{q_1, q_2, q_3\}$, we rewrite the necessary condition in another set of shape variables μ itself and ρ ,

$$\mu = I^{1/2} \left(\frac{1}{r_{12}} + \frac{1}{r_{23}} + \frac{1}{r_{31}} \right), \quad \rho = \frac{I^{3/2}}{r_{12} r_{23} r_{31}}, \quad (6)$$

that are manifestly invariant under the permutations. Since, we are considering $\mu = \text{constant}$ orbits, variables μ and ρ make our proof easy. This expression is given in section 4. The proof of the Saari’s conjecture is given in the section 5. In section 6, we give discussions.

2. Equations of motion

In this section, we summarize the equations of motion for $\alpha = 1$ in terms of size, rotation and shape. We don't assume $\mu = \text{constant}$ in this section.

Let $q_k \in \mathbb{C}$ be the position and mass $m_k = 1$ for $k = 1, 2, 3$. We take the center of mass frame, $\sum q_k = 0$. The Lagrangian is given by,

$$L = \frac{1}{2} \sum \left| \frac{dq_k}{dt} \right|^2 + U. \quad (7)$$

We take the shape variable $\zeta \in \mathbb{C}$ in (4). This variable is invariant under the scaling and rotation, $q_k \rightarrow \lambda e^{i\theta} q_k$ with $\lambda, \theta \in \mathbb{R}$. Thus, ζ depends only on shape. Let us define $\xi_k = q_k / (q_2 - q_1)$. Then, we have,

$$\xi_1 = -\frac{1}{2} - \frac{\zeta}{3}, \quad \xi_2 = \frac{1}{2} - \frac{\zeta}{3}, \quad \xi_3 = \frac{2\zeta}{3}. \quad (8)$$

Since, the triangle $q_1 q_2 q_3$ and $\xi_1 \xi_2 \xi_3$ are similar and have the same orientation, we have two variables $I \geq 0$ and $\theta \in \mathbb{R}$, such that

$$q_k = \sqrt{I} e^{i\theta} \frac{\xi_k}{\sqrt{\sum |\xi_\ell|^2}}. \quad (9)$$

We take I , θ and ζ for dynamical variables. In the following, we identify $\zeta = x + iy$ and $\mathbf{x} = (x, y) \in \mathbb{R}^2$. By direct calculations, we obtain the Lagrangian

$$L = \frac{\dot{I}^2}{8I} + \frac{I}{2} \left(\dot{\theta} + \frac{\frac{4}{3} \mathbf{x} \wedge \dot{\mathbf{x}}}{1 + \frac{4}{3} |\mathbf{x}|^2} \right)^2 + \frac{I}{2} \frac{\frac{4}{3} |\dot{\mathbf{x}}|^2}{(1 + \frac{4}{3} |\mathbf{x}|^2)^2} + \frac{\mu(\mathbf{x})}{\sqrt{I}}. \quad (10)$$

Here, $\dot{\cdot}$ represents time derivative, $\mathbf{x} \wedge \dot{\mathbf{x}} = x\dot{y} - y\dot{x}$ and

$$\mu(\mathbf{x}) = \sqrt{\frac{1}{2} + \frac{2}{3} |\mathbf{x}|^2} \left(1 + \frac{1}{\sqrt{(x-1/2)^2 + y^2}} + \frac{1}{\sqrt{(x+1/2)^2 + y^2}} \right). \quad (11)$$

Since, θ is cyclic, the angular momentum C is constant of motion,

$$C = \frac{\partial L}{\partial \dot{\theta}} = I \left(\dot{\theta} + \frac{\frac{4}{3} \mathbf{x} \wedge \dot{\mathbf{x}}}{1 + \frac{4}{3} |\mathbf{x}|^2} \right). \quad (12)$$

Therefore, the total energy E is given by

$$E = \frac{\dot{I}^2}{8I} + \frac{C^2}{2I} + \frac{I}{2} \frac{\frac{4}{3} |\dot{\mathbf{x}}|^2}{(1 + \frac{4}{3} |\mathbf{x}|^2)^2} - \frac{\mu(\mathbf{x})}{\sqrt{I}}. \quad (13)$$

The three terms in the kinetic energy are kinetic energy for the size change, for the rotation and for the shape change respectively. The equation of motion for I yields Lagrange-Jacobi identity, $\ddot{I} = 4E + 2U$. From this equation, we get the following "Saari's relation" [10],

$$\frac{d}{dt} \left(\frac{I^2}{2} \frac{\frac{4}{3} |\dot{\mathbf{x}}|^2}{(1 + \frac{4}{3} |\mathbf{x}|^2)^2} \right) = \sqrt{I} \frac{d\mu}{dt}.$$

Using the 'time' variable s defined by

$$\frac{ds}{dt} = \frac{1}{2I} \left(1 + \frac{4}{3} |\mathbf{x}|^2 \right), \quad (14)$$

the Saari's relation is written as

$$\frac{d}{ds} \left(\frac{1}{6} \left| \frac{d\mathbf{x}}{ds} \right|^2 \right) = \sqrt{I} \frac{d\mu}{ds}. \quad (15)$$

The equation of motion for \mathbf{x} in terms of s is

$$\frac{d^2\mathbf{x}}{ds^2} = \frac{4C - \frac{8}{3}\mathbf{x} \wedge \frac{d\mathbf{x}}{ds}}{1 + \frac{4}{3}|\mathbf{x}|^2} \left(\frac{dy}{ds}, -\frac{dx}{ds} \right) + 3\sqrt{I} \frac{\partial\mu}{\partial\mathbf{x}}. \quad (16)$$

Up to here, we didn't assume $\mu = \text{constant}$.

3. Necessary condition

Now, we consider a motion with $\mu = \text{constant}$. By the Saari's relation (15), we have

$$\left| \frac{d\mathbf{x}}{ds} \right| = v \quad (17)$$

with constant $v \geq 0$.

For the case $v = 0$, $d\mathbf{x}/ds = 0$ then $d^2\mathbf{x}/ds^2 = 0$. The equation of motion (16) yields $\partial\mu/\partial\mathbf{x} = 0$. Namely, the motion is homographic and the system stays one of the central configurations.

Let us examine the case $v > 0$. In this case, the point $\mathbf{x}(s)$ moves on the curve $\mu(\mathbf{x})$ with finite speed v . Since the number of points $\partial\mu/\partial\mathbf{x} = 0$ are five, we can always take a finite arc on which $\partial\mu/\partial\mathbf{x} \neq 0$. To keep satisfy $d\mu/ds = 0$, the velocity $d\mathbf{x}/ds$ must be orthogonal to $\partial\mu/\partial\mathbf{x}$, so we have

$$\frac{d\mathbf{x}}{ds} = \frac{\epsilon v}{|\partial\mu/\partial\mathbf{x}|} \left(-\frac{\partial\mu}{\partial y}, \frac{\partial\mu}{\partial x} \right). \quad (18)$$

Here, $\epsilon = \pm 1$ determines the direction of the motion. Then, the acceleration (16) is given by

$$\frac{d^2\mathbf{x}}{ds^2} = \left(\frac{\epsilon v}{(1 + 4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|} \left(4C - \frac{8\epsilon v}{3|\partial\mu/\partial\mathbf{x}|} \mathbf{x} \cdot \frac{\partial\mu}{\partial\mathbf{x}} \right) + 3\sqrt{I} \right) \frac{\partial\mu}{\partial\mathbf{x}}. \quad (19)$$

Thus, the velocity (18) and the acceleration (19) determine the curvature of this orbit

$$\kappa = \frac{1}{1 + 4|\mathbf{x}|^2/3} \left(-\frac{4C}{v} + \frac{8\epsilon}{3|\partial\mu/\partial\mathbf{x}|} \left(\mathbf{x} \cdot \frac{\partial\mu}{\partial\mathbf{x}} \right) \right) - \frac{3\epsilon\sqrt{I}}{v^2} \left| \frac{\partial\mu}{\partial\mathbf{x}} \right|. \quad (20)$$

On the other hand, the curve $\mu(\mathbf{x}) = \text{constant}$ has its own curvature,

$$\kappa = \frac{\epsilon}{|\partial\mu/\partial\mathbf{x}|^3} \left(\left(\frac{\partial\mu}{\partial y} \right)^2 \frac{\partial^2\mu}{\partial x^2} - 2 \frac{\partial\mu}{\partial x} \frac{\partial\mu}{\partial y} \frac{\partial^2\mu}{\partial x \partial y} + \left(\frac{\partial\mu}{\partial x} \right)^2 \frac{\partial^2\mu}{\partial y^2} \right). \quad (21)$$

Equate the two expressions for κ , we have a necessary condition for the motion,

$$\begin{aligned} \sqrt{I} = & -\frac{4\epsilon C v}{3(1 + 4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|} + \frac{8v^2}{9(1 + 4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|^2} \left(\mathbf{x} \cdot \frac{\partial\mu}{\partial\mathbf{x}} \right) \\ & - \frac{v^2}{3|\partial\mu/\partial\mathbf{x}|^4} \left(\left(\frac{\partial\mu}{\partial y} \right)^2 \frac{\partial^2\mu}{\partial x^2} - 2 \frac{\partial\mu}{\partial x} \frac{\partial\mu}{\partial y} \frac{\partial^2\mu}{\partial x \partial y} + \left(\frac{\partial\mu}{\partial x} \right)^2 \frac{\partial^2\mu}{\partial y^2} \right). \end{aligned} \quad (22)$$

This is the condition that any motion with $\mu = \text{constant}$ and $d\mathbf{x}/dt \neq 0$ must satisfy. The equation of motion is invariant under the scale transformation $q_k \rightarrow \lambda q_k$ and $t \rightarrow \lambda^{3/2}t$. This transformation makes $\sqrt{I} \rightarrow \lambda\sqrt{I}$, $C \rightarrow \lambda^{1/2}C$, $\mathbf{x} \rightarrow \mathbf{x}$, $s \rightarrow \lambda^{-1/2}s$,

and $v \rightarrow \lambda^{1/2}v$. Using this invariance, we can take $v = \sqrt{3}$ without losing generality. We write C for ϵC . Then, the necessary condition is

$$\begin{aligned} \sqrt{I} = & -\frac{4C}{\sqrt{3}(1+4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|} + \frac{8}{3(1+4|\mathbf{x}|^2/3)|\partial\mu/\partial\mathbf{x}|^2} \left(\mathbf{x} \cdot \frac{\partial\mu}{\partial\mathbf{x}} \right) \\ & - \frac{1}{|\partial\mu/\partial\mathbf{x}|^4} \left(\left(\frac{\partial\mu}{\partial y} \right)^2 \frac{\partial^2\mu}{\partial x^2} - 2 \frac{\partial\mu}{\partial x} \frac{\partial\mu}{\partial y} \frac{\partial^2\mu}{\partial x \partial y} + \left(\frac{\partial\mu}{\partial x} \right)^2 \frac{\partial^2\mu}{\partial y^2} \right), \end{aligned} \quad (23)$$

and the energy is given by

$$E = \frac{1}{2} \left(\frac{d\sqrt{I}}{dt} \right)^2 + \frac{C^2 + 1}{2I} - \frac{\mu}{\sqrt{I}}. \quad (24)$$

Substituting $d\sqrt{I}/dt = (\partial\sqrt{I}/\partial\mathbf{x}) \cdot (d\mathbf{x}/ds)(ds/dt)$, $d\mathbf{x}/ds$ in (18) and the condition (23) into this expression for the energy, we will obtain the necessary condition expressed only by the shape variable \mathbf{x} . However, the condition (23) in \mathbf{x} turns out to be so complex to treat. In the next section, we will rewrite the condition (23) in a concise form.

4. Invariance of the necessary condition

Since we are considering equal mass case, the theory is invariant under the permutations of positions $\{q_i\}$. The exchange of q_1 and q_2 makes $\zeta \rightarrow -\zeta$ and $\mathbf{x} \rightarrow -\mathbf{x}$. The invariance of the necessary condition (23) is manifest. On the other hand, the cyclic permutation $q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q_1$ makes

$$\zeta \rightarrow \zeta' = \frac{3}{2} \frac{q_1}{q_3 - q_2} = \frac{1}{2} \frac{3/2 + \zeta}{1/2 - \zeta}. \quad (25)$$

The invariance of (23) under this transformation is not manifest. In this section, we will rewrite the necessary condition in a manifestly invariant form.

4.1. Invariants

Under the map (25), the Lagrange points $\zeta = \pm i\sqrt{3}/2$ are fixed and the Euler points $\zeta = -3/2, 0, 3/2$ are cyclically permuted. Let us define $\mu_k = I^{1/2}/r_{ij}$ for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. Expressions by ζ are,

$$\mu_1 = \frac{1}{|\zeta - 1/2|} \sqrt{\frac{1}{2} + \frac{2}{3}|\zeta|^2}, \quad \mu_2 = \frac{1}{|\zeta + 1/2|} \sqrt{\frac{1}{2} + \frac{2}{3}|\zeta|^2}, \quad \mu_3 = \sqrt{\frac{1}{2} + \frac{2}{3}|\zeta|^2}. \quad (26)$$

These three μ_k are also cyclically permuted by (25). Note that the exchange $q_i \leftrightarrow q_j$ makes the exchange $\mu_i \leftrightarrow \mu_j$. Therefore, $\mu = \mu_1 + \mu_2 + \mu_3$ is invariant under the permutations of q_i .

The kinetic energy for the shape change must be invariant. Actually, we can easily check the invariance of

$$\frac{4}{3} \frac{|d\zeta|^2}{(1 + \frac{4}{3}|\zeta|^2)^2}. \quad (27)$$

So, it is natural to treat the space of ζ as a metric space whose distance is given by the equation (27), and the map (25) is the isometric transformation. Actually Wu-Yi Hsiang and Eldar Straume [4, 5], Alain Chenciner and R. Montgomery

[1], R. Montgomery [9], and R. Mockel [7] showed that this metric space is the “shape sphere” and the distance (27) is the distance on the shape sphere. Kenji Hiro Kuwabara and Kiyotaka Tanikawa also noticed that the shape sphere is useful to investigate the equal-mass free-fall problem[6, 12]. The map (25) makes the shape sphere $2\pi/3$ rotation around the axis that connects the two Lagrange points. The map $\zeta \rightarrow -\zeta$ makes π rotation around the axis that connects one of the Euler point (corresponds to $\mathbf{x} = 0$) and one of two-body collision (corresponds to $\mathbf{x} = \infty$).

Let us use the notations in the tensor analysis. We write $\zeta = x^1 + ix^2$, $\mathbf{x} = (x, y) = (x^1, x^2)$ and $\partial_i = \partial/\partial x^i$. The metric tensor g_{ij} and its inverse are

$$g_{ij} = \frac{4}{3} \frac{\delta_{ij}}{(1 + \frac{4}{3}|\mathbf{x}|^2)^2}, \quad (g_{ij})^{-1} = g^{ij} = \frac{3}{4} \left(1 + \frac{4}{3}|\mathbf{x}|^2\right)^2 \delta^{ij}, \quad (28)$$

where $\delta_{ij} = \delta^{ij}$ are the Kronecker's delta,

$$\delta_{ij} = \delta^{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \quad (29)$$

Let $|g|$ be the determinant of g_{ij} ,

$$|g| = \det(g_{ij}) = \frac{16}{9} \frac{1}{(1 + \frac{4}{3}|\mathbf{x}|^2)^4}. \quad (30)$$

As mentioned above, the configurational measure μ is invariant. One obvious invariant is the magnitude of the gradient vector of μ . We write

$$|\nabla\mu|^2 = \sum_{i,j} g^{ij} (\partial_i \mu) (\partial_j \mu) = \frac{3}{4} \left(1 + \frac{4}{3}|\mathbf{x}|^2\right)^2 \left| \frac{\partial \mu}{\partial \mathbf{x}} \right|^2. \quad (31)$$

Therefore, the first term of the right hand side of the necessary condition (23) is simply $-2C/|\nabla\mu|$. The other obvious invariant is the Laplacian of μ ,

$$\Delta\mu = \sum_{ij} \frac{1}{\sqrt{|g|}} \partial_i \left(g^{ij} \sqrt{|g|} \partial_j \mu \right) = \frac{3}{4} \left(1 + \frac{4}{3}|\mathbf{x}|^2\right)^2 \frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial \mu}{\partial \mathbf{x}} \quad (32)$$

Now, let us consider the following invariant,

$$\lambda = \sum_{ij} g^{ij} (\partial_i \mu) (\partial_j |\nabla\mu|^2) = \frac{3}{4} \left(1 + \frac{4}{3}|\mathbf{x}|^2\right)^2 \frac{\partial \mu}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{x}} |\nabla\mu|^2. \quad (33)$$

Explicitly performing the differentials, it yields

$$\lambda = 3 \left(1 + \frac{4}{3}|\mathbf{x}|^2\right)^3 \left(\mathbf{x} \cdot \frac{\partial \mu}{\partial \mathbf{x}} \right) \left| \frac{\partial \mu}{\partial \mathbf{x}} \right|^2 + \frac{9}{16} \left(1 + \frac{4}{3}|\mathbf{x}|^2\right)^4 \frac{\partial \mu}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{x}} \left| \frac{\partial \mu}{\partial \mathbf{x}} \right|^2$$

Using this expression, the second and the third terms in the necessary condition (23) is simply expressed as, $\lambda/(2|\nabla\mu|^4) - \Delta\mu/|\nabla\mu|^2$. Thus, the necessary condition is expressed in the following invariant form,

$$\sqrt{I} = -\frac{2C}{|\nabla\mu|} + \frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}. \quad (34)$$

The last obvious invariant what we will use is

$$D\phi = \frac{1}{\sqrt{|g|}} \sum_{i,j} \epsilon^{ij} (\partial_i \mu) (\partial_j \phi) = \frac{3}{4} \left(1 + \frac{4}{3}|\mathbf{x}|^2\right)^2 \frac{\partial \mu}{\partial \mathbf{x}} \wedge \frac{\partial \phi}{\partial \mathbf{x}} \quad (35)$$

for any invariant ϕ . Where, ϵ^{ij} is the Levi-Civita's anti-symmetric symbol,

$$\epsilon^{ij} = \begin{cases} 1 & \text{for } (i, j) = (1, 2), \\ -1 & \text{for } (i, j) = (2, 1), \\ 0 & \text{for } i = j. \end{cases} \quad (36)$$

Then, using equations (14), (18) and (35), we have

$$\frac{d\phi}{dt} = \frac{\epsilon}{I} \frac{D\phi}{|\nabla\mu|}. \quad (37)$$

4.2. Invariant variables

For the Newton potential, it is natural to use the variables r_1 and r_2 defined by (5). Relations between μ_k defined in (26) and r_1, r_2 are

$$\mu_1 = r_1^{-1}\mu_3, \quad \mu_2 = r_2^{-1}\mu_3, \quad \mu_3 = \sqrt{(1 + r_1^2 + r_2^2)/3}. \quad (38)$$

Now, consider the expression for the above invariants $|\nabla\mu|^2$, $\Delta\mu$, λ in terms of r_1 and r_2 . Let us write one of them $\psi(r_1, r_2)$. It is composed by differentials of μ by r_1 or r_2 and products of r_1 and r_2 . Then, the result is composed of terms of rational function of $\sqrt{(1 + r_1^2 + r_2^2)/3}$, r_1 and r_2 , namely μ_3 , μ_3/μ_1 and μ_3/μ_2 . Then, ψ has the following form

$$\psi = f(r_1, r_2) + g(r_1, r_2) \sqrt{\frac{1 + r_1^2 + r_2^2}{3}} = f\left(\frac{\mu_3}{\mu_1}, \frac{\mu_3}{\mu_2}\right) + g\left(\frac{\mu_3}{\mu_1}, \frac{\mu_3}{\mu_2}\right) \mu_3. \quad (39)$$

Here, f and g represent some rational functions. The function ψ is invariant under the permutation of q_i , namely the permutation of μ_i . So, it must be a ratio of some symmetric polynomials of μ_i . Therefore, it must have the following expression

$$\psi = h(\mu, \nu, \rho), \quad (40)$$

where h is a rational function of elementary symmetric polynomials

$$\mu = \mu_1 + \mu_2 + \mu_3, \quad \nu = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad \rho = \mu_1\mu_2\mu_3. \quad (41)$$

Expression in terms of μ_k or in terms of μ, ν, ρ is not unique, since, by the relation (38), there is an identity $\mu_1^{-2} + \mu_2^{-2} + \mu_3^{-2} = 3$. Namely,

$$\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_3^2\mu_1^2 = 3\mu_1^2\mu_2^2\mu_3^2. \quad (42)$$

Therefore, we can eliminate ν , using

$$\nu = \sqrt{2\mu\rho + 3\rho^2}. \quad (43)$$

The expression of $\psi = h(\mu, \sqrt{2\mu\rho + 3\rho^2}, \rho)$ is unique. Thus, the necessary condition will be expressed by a function of invariant shape variables μ and ρ .

Let us express $|\nabla\mu|^2$ by μ and ρ . In terms of r_i , it is

$$|\nabla\mu|^2 = \frac{(1 + r_1^2 + r_2^2)^2}{3} \left(\left(\frac{\partial\mu}{\partial r_1} \right)^2 + \left(\frac{\partial\mu}{\partial r_2} \right)^2 + \frac{r_1^2 + r_2^2 - 1}{r_1 r_2} \frac{\partial\mu}{\partial r_1} \frac{\partial\mu}{\partial r_2} \right). \quad (44)$$

By a direct calculation, we get

$$|\nabla\mu|^2 = \frac{1+r_1^2+r_2^2}{9r_1^4r_2^4} \left(2r_1^4r_2^4(r_1^2+r_2^2) \right. \\ + r_1^4r_2^4(r_1+r_2) - r_1r_2(r_1^7+r_2^7) - r_1^4r_2^4 - 4r_1^3r_2^3(r_1+r_2) \\ + (2r_1^6+r_1^5r_2 - r_1^4r_2^2 - 4r_1^3r_2^3 - r_1^2r_2^4 + r_1r_2^5 + 2r_2^6) \\ \left. + r_1r_2(r_1^3+r_2^3) + 2(r_1^4+r_2^4) - r_1r_2 \right). \quad (45)$$

Substituting $r_1 = \mu_3/\mu_1$ and $r_2 = \mu_3/\mu_2$, we obtain,

$$|\nabla\mu|^2 = \frac{(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_3^2\mu_1^2)}{9\mu_1^6\mu_2^6\mu_3^6} \left(-(\mu_1^7\mu_2^7 + \mu_2^7\mu_3^7 + \mu_3^7\mu_1^7) \right. \\ - \mu_1^4\mu_2^4\mu_3^4(\mu_1^2 + \mu_2^2 + \mu_3^2) - 4\mu_1^4\mu_2^4\mu_3^4(\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1) \\ \left. + 2(\mu_1^8\mu_2^4\mu_3^2 + \dots) + (\mu_1^7\mu_2^4\mu_3^3 + \dots) \right). \quad (46)$$

In the last line, dots in parentheses represent similar 5 terms of permutation of μ_1, μ_2, μ_3 . Then expressing by μ, ν, ρ , we obtain a expression,

$$|\nabla\mu|^2 = \frac{\nu^2 - 2\mu\rho}{9\rho^6} \left(-\nu^7 + 7\mu\nu^5\rho + 2\mu^4\nu^2\rho^2 \right. \\ \left. - 22\mu^2\nu^3\rho^2 - 3\nu^4\rho^2 - 4\mu^5\rho^3 + 24\mu^3\nu\rho^3 + 18\mu\nu^2\rho^3 - 27\mu^2\rho^4 \right). \quad (47)$$

As mentioned above, the expressions (46) and (47) are not unique due to the identity (42). Eliminating ν , we finally get the following unique expression

$$|\nabla\mu|^2 = -\mu^2 + 2\mu^4 + 6\mu\rho - 9\rho^2 - 3(2\mu^2 - \mu\rho + 3\rho^2)\sqrt{2\mu\rho + 3\rho^2}. \quad (48)$$

Thus, we get the expression for $|\nabla\mu|^2$ in manifestly invariant variables μ and ρ .

By a similar way, $\Delta\mu$ in (r_1, r_2) and (μ, ρ) are

$$\Delta\mu = \frac{(1+r_1^2+r_2^2)^2}{3} \left(\frac{1}{r_1} \frac{\partial}{\partial r_1} \left(r_1 \frac{\partial\mu}{\partial r_1} \right) + \frac{1}{r_2} \frac{\partial}{\partial r_2} \left(r_2 \frac{\partial\mu}{\partial r_2} \right) + \frac{r_1^2+r_2^2-1}{r_1r_2} \frac{\partial^2\mu}{\partial r_1\partial r_2} \right), \\ \Delta\mu = \mu + 2\mu^3 + 6\rho - 6\mu\sqrt{2\mu\rho + 3\rho^2}. \quad (49)$$

Similarly, the expressions for λ are

$$\lambda = \frac{(1+r_1^2+r_2^2)^2}{3} \left(\frac{\partial\mu}{\partial r_1} \frac{\partial}{\partial r_1} + \frac{\partial\mu}{\partial r_2} \frac{\partial}{\partial r_2} + \frac{r_1^2+r_2^2-1}{2r_1r_2} \left(\frac{\partial\mu}{\partial r_1} \frac{\partial}{\partial r_2} + \frac{\partial\mu}{\partial r_2} \frac{\partial}{\partial r_1} \right) \right) |\nabla\mu|^2, \\ \lambda = \frac{1}{2} \left(4\mu^3 - 24\mu^5 + 32\mu^7 - 72\mu^2\rho + 660\mu^4\rho + 324\mu\rho^2 \right. \\ + 36\mu^3\rho^2 - 432\rho^3 + 891\mu^2\rho^3 + 2349\mu\rho^4 - 243\rho^5 \\ + 3(24\mu^3 - 60\mu^5 - 156\mu^2\rho + 28\mu^4\rho + 324\mu\rho^2 \\ \left. - 93\mu^3\rho^2 - 216\rho^3 - 27\mu^2\rho^3 + 81\mu\rho^4) \sqrt{2\mu\rho + 3\rho^2} \right). \quad (50)$$

Finally, $(D\rho)^2$ is also invariant under the exchange of q_i , therefore, it has an expression by μ and ρ ,

$$(D\rho)^2 = \frac{(1 + r_1^2 + r_2^2)^4 \left(2(r_1^2 + r_2^2) - (r_1^2 - r_2^2)^2 - 1 \right)}{36r_1^2 r_2^2} \left(\frac{\partial \mu}{\partial r_1} \frac{\partial \rho}{\partial r_2} - \frac{\partial \mu}{\partial r_2} \frac{\partial \rho}{\partial r_1} \right)^2,$$

$$(D\rho)^2 = \frac{\rho^2(2\mu + 3\rho)}{4} \left(- (2\mu + 3\rho)(4\mu^4 + 134\mu\rho - 12\mu^3\rho - 177\rho^2 + 9\mu^2\rho^2) \right. \\ \left. + 2(28\mu^3 + 108\rho - 36\mu^2\rho - 45\mu\rho^2 + 54\rho^3)\sqrt{2\mu\rho + 3\rho^2} \right). \quad (51)$$

5. Proof of the Saari's conjecture

In the previous section, we find the expression for the necessary condition (34) in terms of μ and ρ by (48), (49) and (50). Since, we are assuming $\mu = \text{constant}$, time dependent variable is only ρ . Therefore, $d\sqrt{I}/dt = (\partial\sqrt{I}/\partial\rho)(d\rho/dt)$. Using (37),

$$\left(\frac{d\sqrt{I}}{dt} \right)^2 = \frac{1}{I^2} \frac{(D\rho)^2}{|\nabla\mu|^2} \left(\frac{\partial\sqrt{I}}{\partial\rho} \right)^2.$$

Substituting this expression and the necessary condition (34) into the expression of the energy (24), we obtain the necessary condition for ρ with three parameters E , C and μ ,

$$E = \frac{1}{2I^2} \frac{(D\rho)^2}{|\nabla\mu|^2} \left(\frac{\partial\sqrt{I}}{\partial\rho} \right)^2 + \frac{C^2 + 1}{2I} - \frac{\mu}{\sqrt{I}}. \quad (52)$$

If there is some finite motion with $\mu = \text{constant}$ and non-homographic, this condition must be satisfied by some finite range of ρ . However, since the right hand side of (52) is analytic function of ρ , the condition (52) must be satisfied for all range of ρ .

In the vicinity of $\rho = 0$, we have the expansion of (34)

$$\sqrt{I} = a_0 + a_{1/2}\sqrt{\rho} + a_1\rho + O(\rho^{3/2}), \quad (53)$$

with

$$a_0 = \frac{2(1 - \mu^2 + C\sqrt{-1 + 2\mu^2})}{\mu(1 - 2\mu^2)},$$

$$a_{1/2} = \frac{3\sqrt{2}((-2 + \mu^2)\sqrt{-1 + 2\mu^2} - 2C(-1 + 2\mu^2))}{(1 - 2\mu^2)^2\sqrt{\mu(-1 + 2\mu^2)}},$$

$$a_1 = \frac{3((-2 + \mu^2)(1 + 6\mu^2) - 2C(1 + 7\mu^2)\sqrt{-1 + 2\mu^2})}{\mu^2(-1 + 2\mu^2)^3},$$

and

$$|\nabla\mu|^2 = \mu^2(-1 + 2\mu^2) - 6\sqrt{2}\mu^{5/2} + 6\mu\rho + O(\rho^{3/2}),$$

$$(D\rho)^2 = -4\mu^6\rho^2 + O(\rho^{5/2}).$$

Then we obtain the power series expansion of (52) by $\sqrt{\rho}$ up to the order ρ at $\rho = 0$.

The term of order ρ^0 in (52) determine E . Therefore, this order gives no information for C and μ . The coefficient of order $\sqrt{\rho}$ is

$$0 = \frac{-3\mu^{5/2}\left(1 + C\sqrt{-1 + 2\mu^2}\right)^2\left((-2 + \mu^2) - 2C\sqrt{-1 + 2\mu^2}\right)}{4\sqrt{2}\left(-1 + \mu^2 - C\sqrt{-1 + 2\mu^2}\right)^3}. \quad (54)$$

The solutions C of this equation are,

$$C = -\frac{1}{\sqrt{-1 + 2\mu^2}}, \quad \frac{-2 + \mu^2}{2\sqrt{-1 + 2\mu^2}}. \quad (55)$$

For the case $C = -1/\sqrt{-1 + 2\mu^2}$,

$$\sqrt{I} = \frac{2\mu}{-1 + 2\mu^2} + \frac{3\sqrt{2}\mu^{3/2}}{(-1 + 2\mu^2)^2}\sqrt{\rho} + \frac{9(1 + 2\mu^2)}{(-1 + 2\mu^2)^3}\rho + O(\rho^{3/2}), \quad (56)$$

and the order ρ^1 coefficient in the equation (52) is

$$0 = -\frac{9\mu(-2 + \mu^2)}{16(-1 + 2\mu^2)}. \quad (57)$$

While the right hand side is always negative since $\mu = \sqrt{(1 + r_1^2 + r_2^2)/3}(1 + 1/r_1 + 1/r_2) \geq 3$. For the case $C = (-2 + \mu^2)/(2\sqrt{-1 + 2\mu^2})$, the coefficient $a_{1/2}$ vanish,

$$\sqrt{I} = \frac{\mu}{4(-1 + 2\mu^2)} - \frac{3(-2 + \mu^2)}{4(-1 + 2\mu^2)^3}\rho + O(\rho^{3/2}), \quad (58)$$

and the coefficient of order ρ^1 in the equation (52) is

$$0 = \frac{3\mu(-2 + \mu^2)}{4(-1 + 2\mu^2)}. \quad (59)$$

While the right hand side is always positive for $\mu \geq 3$.

Thus, there is no parameters C and μ that satisfies the necessary condition (52). This completes the proof for the Saari's homographic conjecture.

6. Discussions

We have proved the Saari's conjecture for equal-mass planar three-body problem under the Newton gravity.

The symmetry under the permutation of the positions $\{q_1, q_2, q_3\}$ has a crucial role for our method. For equal mass and Newton potential case, the necessary condition (34) is a symmetric rational function of μ_1, μ_2 and μ_3 . Thus, it is a function of μ and ρ as in equation (40). This makes our proof simple.

The next step will be the case with general mass ratio and general homogeneous potential $U = \sum m_i m_j / r_{ij}^\alpha$, $\alpha > 0$. For this case, however, an invariant function under the permutation for suffix of bodies will not have a simple form of manifestly invariant variables such as μ and ρ . We hope, someday, someone may find a proof for the conjecture for general mass ratio under the Newton potential in some extension of our method. On the other hand, we are afraid that it is hard to extend our method to general α . We would have to find a completely new method for general α .

Acknowledgments

This research of one of the author T. Fujiwara has been supported by Grand-in-Aid for Scientific Research 23540249 JSPS.

References

- [1] A. Chenciner and R. Montgomery, *A remarkable periodic solution of the three-body problem in the case of equal masses*, Annals of Mathematics, **152**, 881–901, 2000.
- [2] F. Diacu, T. Fujiwara, E. Pérez-Chavela and M. Santoprete, *Saari's homographic conjecture of the three-body problem*, Transactions of the American Mathematical Society, **360**, 12, 6447–6473, 2008.
- [3] T. Fujiwara, H. Fukuda, H. Ozaki and T. Taniguchi, *Saari's homographic conjecture for planar equal-mass three-body problem under a strong force potential*, J. Phys. A: Math. Theor. **45** (2012) 045208.
- [4] W. Y. Hsiang and E. Straume, *Kinematic geometry of triangles with given mass distribution*, PAM-636 (1995), Univ. of Calif., Berkeley.
- [5] W. Y. Hsiang and E. Straume, *Kinematic geometry of triangles and the study of the three-body problem*, arXiv:math-ph/0608060, 2006.
- [6] K. H. Kuwabara and K. Tanikawa, *A new set of variables in the three-body problem*, Publications of the Astronomical Society of Japan Vol. 62, pp 1–7, 2010.
- [7] R. Moeckel, *Shooting for the eight – a topological existence proof for a figure-eight orbit of the three-body problem*, <http://www.math.umn.edu/~rmoeckel/research/FigureEight12.pdf>, 2007.
- [8] R. Moeckel and R. Montgomery, private communications.
- [9] R. Montgomery, *Infinitely Many Syzygies*, Archive for Rational Mechanics and Analysis, 2002, Volume 164, Number 4, pp 311–340.
- [10] D. Saari, *Collisions, rings, and other Newtonian N-body problems*, CBMS Regional Conference Series in Mathematics, number 104, American Mathematical Society, 2005.
- [11] D. Saari, *Some ideas about the future of Celestial Mechanics*, Conf. Saarifest (Guanajuato, Mexico, 8 April), 2005. See, Donald Saari, *Reflections on my conjecture, and several new ones* (<http://math.uci.edu/~dsaari/conjecture-revisited.pdf>)
- [12] K. Tanikawa and K. Kuwabara, *The planar three-body problem with angular momentum*, in 'Resonances, Stabilization, and Stable Chaos in Hierarchical Triple Systems', pp. 71–76, Eds. V.V. Orlov and A.V. Rubinov, Proceedings of a workshop held in Saint Petersburg, Russia, 26–29 August 2007, Saint Petersburg University, Saint Petersburg, 2008.